

Tail asymptotics for the supercritical Galton–Watson process in the heavy-tailed case ¹

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Abstract

As well known, for a supercritical Galton–Watson process Z_n whose offspring distribution has mean $m > 1$, the ratio $W_n := Z_n/m^n$ has a.s. limit, say W . We study tail behaviour of the distributions of W_n and W in the case where Z_1 has heavy-tailed distribution, that is, $\mathbb{E}e^{\lambda Z_1} = \infty$ for every $\lambda > 0$. We show how different types of distributions of Z_1 lead to different asymptotic behaviour of the tail of W_n and W . We describe the most likely way how large values of the process occur.

Keywords: supercritical Galton–Watson process, martingale limit, large deviations, heavy-tailed distribution, subexponential distribution, square-root insensitive distribution, Weibull type distribution

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1 Introduction

Let Z_n be a supercritical Galton–Watson process with $Z_0 = 1$, $m := \mathbb{E}Z_1 > 1$. By definition,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)},$$

where $\xi_i^{(n)}$, $i, n = 0, 1, \dots$, are independent identically distributed random variables with distribution F on $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$; by $\overline{F}(x)$ we denote the tail of F , $\overline{F}(x) := \mathbb{P}\{\xi > x\}$.

Put $W_n := Z_n/m^n$. As well known (see, e.g., [2, Theorem 1.6.1]) $W_n \rightarrow W$ a.s. as $n \rightarrow \infty$. If $\mathbb{E}\xi \log \xi < \infty$ then $\mathbb{E}W = 1$, so $\mathbb{P}\{W > 0\} > 0$, see [2, Theorem 1.10.1].

Our goal is to consider asymptotic probabilities for the martingale sequence $\{W_n\}$ and for its limit W . More precisely, we are going to find asymptotics for $\mathbb{P}\{W_n > x\}$ as $x \rightarrow \infty$ in the whole range of $n \geq 1$.

The tail-behaviour of the martingale limit is one of the classical problems in the theory of supercritical Galton–Watson processes. The study of $\mathbb{P}\{W > x\}$ has been initiated by Harris [14] who showed that if ξ is bounded, then

$$\log \mathbb{E}e^{uW} = u^\gamma H(u) + O(1) \quad \text{as } u \rightarrow \infty,$$

where H is a positive multiplicatively periodic function and γ is defined by the equality $m^\gamma = \max\{k : \mathbb{P}\{\xi = k\} > 0\}$. This information on the generating

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function can be translated into asymptotics of tail-probabilities. It was done by Biggins and Bingham [4]:

$$\log \mathbb{P}\{W > x\} \sim -x^{\gamma/(\gamma-1)} M(x), \quad (1)$$

where M is also a positive multiplicatively periodic function; hereinafter we write $f(x) \sim g(x)$ as $x \rightarrow \infty$ if $f(x)/g(x) \rightarrow 1$. Bingham and Doney [5, 6] found asymptotics for $\mathbb{P}\{W > x\}$ in the case when ξ is regularly varying with non-integer index $\alpha < -1$ (for the case of integer α see De Meyer [8]). In [4] one can find similar to (1) results for the left tail of W in the case, when the minimum offspring size is at least 2. Fleischmann and Wachtel [11, 12] found exact (without logarithmic scaling) asymptotics for $\mathbb{P}\{W_n \in (0, x)\}$ and $\mathbb{P}\{W \in (0, x)\}$ as $x \rightarrow 0$. These two papers give a complete description of the asymptotic behaviour of the left tail of W . It is possible to adapt the method from [12] to upper deviation problems for processes with polynomial offspring generating functions. As a result one gets exact asymptotics for $\mathbb{P}\{W > x\}$ as $x \rightarrow \infty$, see Remark 3 in [12].

In all the papers mentioned above, the proofs were based on the fact that $\varphi(u) := \mathbb{E}e^{-uW}$ satisfies the Poincare functional equation, $\varphi(mu) = f(\varphi(u))$, where f stands for the offspring generating function. In the present paper we do not use that equation. Instead, we apply probabilistic techniques for sums of independent identically distributed variables and for Galton–Watson processes with heavy tails which were developed in recent years.

We work with the following classes of distributions.

Distribution of a random variable ξ is called *heavy-tailed* if $\mathbb{E}e^{\lambda\xi} = \infty$ for every $\lambda > 0$.

We say that a distribution F on \mathbb{R} is *dominated varying*, and write $F \in \mathcal{D}$, if

$$\sup_x \frac{\overline{F}(x/2)}{\overline{F}(x)} < \infty. \quad (2)$$

A distribution F on \mathbb{R} is called *intermediate regularly varying* if

$$\lim_{\varepsilon \downarrow 0} \liminf_{x \rightarrow \infty} \frac{\overline{F}(x(1+\varepsilon))}{\overline{F}(x)} = 1.$$

Note that any regularly varying distribution is intermediate regularly varying. Any intermediate regularly varying distribution is dominated varying.

For any positive function $h(x) \rightarrow \infty$, we say that F is *h -insensitive* if $\overline{F}(x+h(x)) \sim \overline{F}(x)$ as $x \rightarrow \infty$. A distribution F is intermediate regularly varying if and only if F is h -insensitive for any positive function h such that $h(x) = o(x)$ as $x \rightarrow \infty$; in other words, if F is $o(x)$ -insensitive (see [13, Theorem 2.47]).

We say that a distribution F on \mathbb{R}^+ with mean m is *strong subexponential*, and write $F \in \mathcal{S}^*$, if

$$\int_0^x \overline{F}(x-y) \overline{F}(y) dy \sim 2m \overline{F}(x) \quad \text{as } x \rightarrow \infty.$$

Among strong subexponential distributions are intermediate regularly varying, log-normal and Weibull with parameter $\beta < 1$. Any dominated varying distribution is in \mathcal{S}^* if it is long-tailed, that is, constant-insensitive.

A distribution F is called *rapidly varying* if, for any $\varepsilon > 0$,

$$\overline{F}(x(1 + \varepsilon)) = o(\overline{F}(x)) \quad \text{as } x \rightarrow \infty.$$

Clearly this class includes Weibull distributions $\overline{F}(x) = e^{-x^\beta}$ with parameter $\beta > 0$. The log-normal distribution is also rapidly varying. This class does not include intermediate regularly varying distributions.

Theorem 1. *Let F be dominated varying distribution such that, for some $\delta > 0$ and $c < \infty$,*

$$\overline{F}(xy) \leq c\overline{F}(x)/y^{1+\delta} \quad \text{for all } x, y > 1. \quad (3)$$

Then there exist constants $c_1 > 0$ and $c_2 < \infty$ such that

$$c_1\overline{F}(x) \leq \mathbb{P}\{W_n > x\} \leq c_2\overline{F}(x) \quad \text{for all } x, n. \quad (4)$$

If, in addition, F is intermediate regularly varying distribution, then, uniformly in n ,

$$\mathbb{P}\{W_n > x\} \sim \sum_{i=0}^{n-1} m^i \overline{F}(m^{i+1}x) \quad \text{as } x \rightarrow \infty. \quad (5)$$

In particular,

$$\mathbb{P}\{W_n > x\} \sim \sum_{i=0}^{\infty} m^i \overline{F}(m^{i+1}x) \quad \text{as } x, n \rightarrow \infty, \quad (6)$$

and

$$\mathbb{P}\{W > x\} \sim \sum_{i=0}^{\infty} m^i \overline{F}(m^{i+1}x) \quad \text{as } x \rightarrow \infty. \quad (7)$$

As follows from the proof of Lemma 9,

$$\mathbb{P}\left\{\max_{i \leq Z_k} \xi_i^{(k)} \geq m^{k+1}x\right\} \sim m^k \overline{F}(m^{k+1}x) \quad \text{as } x \rightarrow \infty$$

and the summand $m^k \overline{F}(m^{k+1}x)$ in (5)–(7) describes the probability of the existence of a very productive particle in the k -th generation. We can informally restate (5)–(7) as follows

$$\{W_n > x\} \approx \bigcup_{k=0}^{n-1} \left\{ \max_{i \leq Z_k} \xi_i^{(k)} \geq m^{k+1}x \right\}$$

and

$$\{W > x\} \approx \bigcup_{k=0}^{\infty} \left\{ \max_{i \leq Z_k} \xi_i^{(k)} \geq m^{k+1}x \right\}.$$

Moreover, if $\overline{F}(x)$ is regularly varying with index $\alpha < -1$ then, uniformly in n ,

$$\mathbb{P}\left\{\max_{i \leq Z_k} \xi_i^{(k)} \geq m^{k+1}x \mid W_n > x\right\} \rightarrow \frac{m^{-(\alpha-1)k}}{\sum_{j=0}^{n-1} m^{-(\alpha-1)j}} \quad \text{as } x \rightarrow \infty.$$

In the limit $n \rightarrow \infty$ we get the geometric distribution with the parameter $m^{-(\alpha-1)}$. Therefore, atypically big values of the limit W are caused by a very productive particle which lives in one of the initial generations, and the number of this generation is random with the geometric distribution mentioned above.

If we assume the second moment of ξ finite then we may relax the regularity condition on F , namely we may consider distributions which are not necessarily intermediate regularly varying as was assumed in Theorem 1.

Theorem 2. *Let F be dominated varying distribution and the condition (3) hold. If $\mathbb{E}\xi^2 < \infty$ and F is x^γ -insensitive distribution for some $\gamma > 1/2$, then the asymptotics (5), (6) and (7) hold.*

We next turn to the case of Weibull-type offspring distributions.

Theorem 3. *Let $\bar{F}(x) = e^{-R(x)}$ where $R(x)$ is regularly varying with index $\beta \in (0, 1)$. Additionally assume that $F \in \mathcal{S}^*$. Then, for every $\varepsilon > 0$,*

$$(1 + o(1))\bar{F}((m + \varepsilon)x) \leq \mathbb{P}\{W_n > x\} \leq (1 + o(1))\bar{F}((m - \varepsilon)x)$$

as $x \rightarrow \infty$ uniformly in n .

If $\beta < \frac{3-\sqrt{5}}{2} \approx 0.382$ then $\mathbb{P}\{W_n > x\} \sim \bar{F}(mx)$ as $x \rightarrow \infty$ uniformly in n and $\mathbb{P}\{W > x\} \sim \bar{F}(mx)$ as $x \rightarrow \infty$.

If $\beta < 1/2$ and, in addition, for some $c_1 < \infty$,

$$R(k) - R(k-1) \leq c_1 \frac{R(k)}{k}, \quad k \geq 1, \quad (8)$$

then $\mathbb{P}\{W_n > x\} \sim \mathbb{P}\{W > x\} \sim \bar{F}(mx)$ as $x \rightarrow \infty$ uniformly in n .

Let us make a remark on Weibull-type offspring distributions which are not \sqrt{x} -insensitive. If $\mathbb{P}\{\xi > x\} \sim e^{-x^\beta}$ with some $\beta \in (1/2, 1)$, then

$$\mathbb{P}\{W_n > x\} \geq \exp\left\{-(mx)^\beta + \frac{\beta^2 \sigma_n^2}{2}(mx)^{2\beta-1}(1 + o(1))\right\}, \quad n \geq 2, \quad (9)$$

and

$$\mathbb{P}\{W > x\} \geq \exp\left\{-(mx)^\beta + \frac{\beta^2 \sigma^2}{2}(mx)^{2\beta-1}(1 + o(1))\right\}. \quad (10)$$

Here $\sigma_n^2 := \mathbb{E}(W_n - 1)^2$ and $\sigma^2 := \mathbb{E}(W - 1)^2$. These bounds imply that, in contrast to the case $\beta < 1/2$, $\mathbb{P}\{W_n > x\} \gg \bar{F}(mx)$ for all $n \geq 2$. At the end of Section 3 we give arguments for (10).

In Theorem 3 we have, uniformly in n ,

$$\{W_n > x\} \approx \{\xi_1^{(0)} > mx\}.$$

Thus, large values of all W_n are caused by a correspondingly large first generation.

The importance of initial generations for deviation probabilities can be explained by the multiplicative structure of supercritical Galton-Watson processes. As a consequence of this fact, it is ‘cheaper’ to have some special type of behaviour at the very beginning of the process. In Theorems 1 and 3 we see a quite strong localisation: only finite number of generations is important. There are some examples in the literature where a weaker form of the localisation occurs. In the case of lower

deviations which were studied in [11, 12], the optimal strategy looks as follows: In order to have $\{Z_n = k_n\}$ with some $k_n = o(m^n)$ every particle in first a_n generations should have exactly $\mu := \min\{k : \mathbb{P}\{\xi = k\} > 0\}$ children. (Here we assume, for simplicity, that $\xi \geq 1$.) In all later generations we let Z_k grow without any restriction, i.e., geometrically with the rate m . Since we want to get k_n particles in the n -th generation, a_n should satisfy $\mu^{a_n} m^{n-a_n} \approx k_n$. Recalling that $k_n = o(m^n)$, we see that the number of generations with non-typical behaviour tends to infinity. A similar strategy is behind asymptotics for $\mathbb{P}\{W < \varepsilon\}$ as $\varepsilon \rightarrow 0$ and behind asymptotics for upper deviations of processes with polynomial generating functions. This localisation effect for Galton-Watson processes with vanishing limit, that is, Z_n conditioned on $\{W < \varepsilon\}$ with $\varepsilon \rightarrow 0$, was recently studied by Berestycki, Gantert, Mörters and Sidorova [3]. They showed that the genealogical tree coincides up to a certain generation with the regular μ -ary tree.

It turns out that this type of optimal strategies is not universal for supercritical Galton-Watson processes. The next result shows that if the offspring distribution has only the first power moment, then large values of W_n and W can be produced by the middle part of the tree.

Theorem 4. *Assume that $\mathbb{E}\xi \log \xi < \infty$ and $\bar{F}(x)$ is regularly varying with index -1 . Then, uniformly in $n \geq 1$,*

$$\mathbb{P}\{W_n > x\} \sim \sum_{i=0}^{n-1} m^i \bar{F}(m^{i+1}x) \sim \frac{1}{m \log m} x^{-1} \int_x^{m^n x} \bar{F}(u) du \quad \text{as } x \rightarrow \infty. \quad (11)$$

For the limit W we have

$$\mathbb{P}\{W > x\} \sim \sum_{i=0}^{\infty} m^i \bar{F}(m^{i+1}x) \sim \frac{1}{m \log m} x^{-1} \int_x^{\infty} \bar{F}(u) du \quad \text{as } x \rightarrow \infty. \quad (12)$$

Relation (12) is a refinement of Theorem 1.4 in [5] where the following was proved: If $\mathbb{E}\{Z_1; Z_1 > x\} \sim L(x)$ for some slowly varying function L satisfying $\int_1^{\infty} \frac{L(x)}{x} dx < \infty$, then

$$\mathbb{E}\{W; W > x\} \sim \frac{1}{m \log m} \int_x^{\infty} \frac{L(y)}{y} dy.$$

Noting that $\bar{F}(x) = o\left(x^{-1} \int_x^{\infty} \bar{F}(u) du\right)$, we conclude from Theorem 4 that, for every $N \geq 1$,

$$\sum_{i=0}^N m^i \bar{F}(m^{i+1}x) = o(\mathbb{P}(W > x)) \quad \text{as } x \rightarrow \infty.$$

This means that 'big jumps' in any fixed number of generations do not affect large values of W . Furthermore, the main contribution to $\sum_{i=0}^{\infty} m^i \bar{F}(m^{i+1}x)$ (and therefore to $\mathbb{P}\{W > x\}$) comes from indices i such that the ratio $\frac{\int_{m^i x}^{\infty} \bar{F}(u) du}{\int_x^{\infty} \bar{F}(u) du}$ is bounded away from 0 and 1. For finite values of n we have three different regimes depending on the relation between n and x . We illustrate them by the following example.

Example 1.1. Assume that $\bar{F}(x) \sim x^{-1} \log^{-p-1} x$ with some $p > 1$. Then

$$L(x) := \int_x^{\infty} \bar{F}(y) dy \sim \frac{1}{p} \log^{-p} x \quad \text{as } x \rightarrow \infty.$$

Therefore,

$$\mathbb{P}\{W > x\} \sim \frac{1}{m \log m} x^{-1} L(x) \sim \frac{1}{pm \log m} x^{-1} \log^{-p} x. \quad (13)$$

Consider now finite values of n .

First, if n and x are such that $\frac{n}{\log x} \rightarrow \infty$, then, according to (11),

$$\begin{aligned} \mathbb{P}\{W_n > x\} &\sim \frac{1}{m \log m} x^{-1} (L(x) - L(m^n x)) \\ &\sim \frac{1}{m \log m} x^{-1} L(x) \sim \frac{1}{pm \log m} x^{-1} \log^{-p} x. \end{aligned}$$

Comparing this with (13), we see that asymptotics of $\mathbb{P}\{W_n > x\}$ and $\mathbb{P}\{W > x\}$ are equal in this case.

Second, if n and x are such that $\frac{n}{\log x} \rightarrow t \in (0, \infty)$, then

$$L(m^n x) \sim \frac{1}{p} (\log x + n \log m)^{-p} \sim \frac{1}{p} \log^{-p} x (1 + t \log m)^{-p}.$$

Consequently,

$$\mathbb{P}\{W_n > x\} \sim \frac{1}{pm \log m} x^{-1} \log^{-p} x (1 - (1 + t \log m)^{-p}).$$

Here we see that $\mathbb{P}\{W_n > x\}$ and $\mathbb{P}\{W > x\}$ are still of the same order, but the constants are different.

Third, if $\frac{n}{\log x} \rightarrow 0$ then, noting that $\log y \sim \log x$ uniformly in $y \in [x, m^n x]$, we have

$$\begin{aligned} \mathbb{P}\{W_n > x\} &\sim \frac{1}{m \log m} x^{-1} \int_x^{m^n x} \frac{dy}{y \log^{p+1} y} \\ &\sim \frac{1}{m \log m} \frac{1}{x \log^{p+1} x} \int_x^{m^n x} \frac{dy}{y} \sim n \bar{F}(mx). \end{aligned}$$

Therefore, $\mathbb{P}\{W_n > x\}$ is much smaller than $\mathbb{P}\{W > x\}$ for these values of n .

The problem of describing tail asymptotics for supercritical Galton–Watson process is closely related to the problem of tail behaviour for randomly stopped sum S_τ where the random number τ of summands has the same distribution as the summands ξ 's have. For random sums, the only case well studied is the case where the distribution tail of τ is much lighter than that of ξ , see [10]; in this case the typical answer is $\mathbb{P}\{S_\tau > x\} \sim \mathbb{E}\tau \mathbb{P}\{\xi > x\}$ as $x \rightarrow \infty$. The present study may be considered as a step towards general problem for randomly stopped sums where the tails of the stopping time τ and of the summand ξ are comparable.

The rest of the paper is organised as follows. Section 2 is devoted to related upper bounds for the distribution tails of sums with zero drift in the of large deviation zone. Later on in Section 4 they serve for deriving upper bounds for $\mathbb{P}\{W_n > x\}$; more precisely, we reduce the problem of finding the asymptotic behaviour of $\mathbb{P}\{W_n > x\}$ to that for $\mathbb{P}\{W_N > x\}$ with some fixed N . Also, upper bounds of Section 2 help to compute asymptotics for $\mathbb{P}\{W_N > x\}$ for every fixed N . Lower bounds for the distribution tail of the number of descendants in the n th generation are given in Section 3. In Section 6 we provide final proofs of Theorems 1, 2 and 3. Finally, for Theorem 4 where only the first moment is finite, our approach based on describing and computing the most likely events leading to large deviations of W_n doesn't work. Here we propose an analytic method adapted from [17], see Section 7.

2 Preliminary results for sums

We repeatedly make use of the following result which is a version of Theorem 2(i) in [10] with exactly the same proof. In what follows η_1, η_2, \dots are independent random variables with common distribution G and $T_n := \eta_1 + \dots + \eta_n$.

Proposition 5. *Let the distribution G have negative mean $a := \mathbb{E}\eta_1 < 0$. If $G \in \mathcal{S}^*$ then*

$$\mathbb{P}\{T_n > x\} \leq (1 + o(1))n\overline{G}(x)$$

as $x \rightarrow \infty$ uniformly in n .

The latter proposition helps to deduce exact asymptotics for $\mathbb{P}\{T_n > x\}$ in the case of zero mean if $x/n > c > 0$. If $x = o(n)$ then Proposition 5 is not useful for estimation of $\mathbb{P}\{T_n > x\}$ in the case of zero mean. So, in the following two propositions we derive rough upper bounds for the large deviation probabilities for sums with zero mean; these rough bounds will be appropriate for our purposes. The first proposition is devoted to distributions of regularly varying type while the second one is devoted to Weibullian type distributions. Deriving rather rough bounds, we relax conditions on distribution of jumps comparing to the asymptotic results of [9, Theorems 8.1 and 8.3] and [7, Theorems 3.1.1, 4.1.2 and 5.2.1].

Proposition 6. *Let $\mathbb{E}\eta_1 = 0$, $\mathbb{E}\{\eta_1^2; \eta_1 \leq 0\} < \infty$ and G be a dominated varying distribution.*

If, for some $\delta \in (0, 1)$,

$$\mathbb{E}\{\eta^{1+\delta}; \eta > 0\} < \infty, \tag{14}$$

then, for every $\delta' \in (0, \delta)$, there exists $c < \infty$ such that $\mathbb{P}\{T_n > x\} \leq cn\overline{G}(x)$ for all $x > 0$ and $n \leq x^{1+\delta'}$.

If, for some $\delta > 0$,

$$\mathbb{E}\{\eta^{2+\delta}; \eta > 0\} < \infty, \tag{15}$$

then there exists $c < \infty$ such that $\mathbb{P}\{T_n > x\} \leq cn\overline{G}(x)$ for all $x > 0$ and $n \leq x^2/c \log x$ (or equivalently, $x \geq c\sqrt{n \log n}$).

Proof. Let $R(x)$ be the hazard function for G , that is, $\overline{G}(x) = e^{-R(x)}$. First prove that dominated variation yields, for some $C < \infty$, the upper bound

$$R(x) \leq C + C \log x, \quad x \geq 1. \tag{16}$$

Indeed, there exists $c < \infty$ such that $\overline{G}(x/2) \leq e^c \overline{G}(x)$ for all x . Equivalently, $R(x/2) \geq R(x) - c$ which implies $R(x2^{-n}) \geq R(x) - cn$. For $n(x) := [\log_2 x] + 1$ we get

$$\begin{aligned} R(1) &\geq R(x2^{-n(x)}) \\ &\geq R(x) - cn(x) \geq R(x) - c \log_2 x - c \end{aligned}$$

and the upper bound (16) follows.

For every $y < x$, we may estimate the tail distribution of the sum as follows:

$$\begin{aligned}\mathbb{P}\{T_n > x\} &\leq \mathbb{P}\{T_n > x, \eta_k > y \text{ for some } k \leq n\} + \mathbb{P}\{T_n > x, \eta_k \leq y \text{ for all } k \leq n\} \\ &\leq n\overline{G}(y) + e^{-\lambda x}(\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq y\})^n,\end{aligned}\quad (17)$$

for every $\lambda > 0$, by the exponential Chebyshev inequality. Fix an $\varepsilon \in (0, 1)$. Take $y := \varepsilon x$ and $\lambda := 2R(x)/x$. Then $e^{-\lambda x} = \overline{G}(x)e^{-R(x)}$ and

$$\mathbb{P}\{T_n > x\} \leq n\overline{G}(y) + \overline{G}(x)e^{-R(x)}(\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq y\})^n.$$

Let us estimate the latter truncated exponential moment:

$$\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq y\} = \mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq 1/\lambda\} + \mathbb{E}\{e^{\lambda\eta_1}; 1/\lambda < \eta_1 \leq y\}. \quad (18)$$

Since $e^u \leq 1 + u + 2u^2$ for all $u \leq 1$,

$$\begin{aligned}\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq 1/\lambda\} &\leq 1 + \lambda\mathbb{E}\{\eta_1; \eta_1 \leq 1/\lambda\} + 2\lambda^2\mathbb{E}\{\eta_1^2; \eta_1 \leq 1/\lambda\} \\ &\leq 1 + 2\lambda^2\mathbb{E}\{\eta_1^2; \eta_1 \leq 1/\lambda\},\end{aligned}\quad (19)$$

owing to the mean zero for η_1 .

Consider the case of finite second moment where we get

$$\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq 1/\lambda\} \leq 1 + c_1\lambda^2. \quad (20)$$

Further,

$$\begin{aligned}\mathbb{E}\{e^{\lambda\eta_1}; 1/\lambda < \eta_1 \leq y\} &\leq e^{\lambda y}\overline{G}(1/\lambda) \\ &\leq e^{\lambda y}\mathbb{E}\{\eta^{2+\delta}; \eta > 0\}\lambda^{2+\delta},\end{aligned}$$

by the condition (15) and the Chebyshev inequality. Choose $\varepsilon > 0$ so small that $\varepsilon C < \delta/4$. Then the upper bound (16) yields, for some $c_2 < \infty$,

$$e^{\lambda y} = e^{\varepsilon x 2R(x)/x} \leq c_2 x^{\delta/2}$$

and consequently

$$\mathbb{E}\{e^{\lambda\eta_1}; 1/\lambda < \eta_1 \leq y\} \leq c_3\lambda^2. \quad (21)$$

Together with (20) it implies that

$$\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq y\} \leq 1 + c_4 R^2(x)/x^2 \leq e^{c_4 R^2(x)/x^2},$$

for some $c_4 < \infty$. Hence,

$$\begin{aligned}\mathbb{P}\{T_n > x\} &\leq n\overline{G}(y) + \overline{G}(x)e^{-R(x)}e^{c_4 n R^2(x)/x^2} \\ &\leq n\overline{G}(y) + \overline{G}(x)e^{-R(x) + R(x)(c_5 n \log x/x^2)},\end{aligned}$$

for some $c_5 < \infty$, due to (16). So, in the case of finite $2 + \delta$ moment, the proposition conclusion follows for $n \leq x^2/c_5 \log x$ if we take into account (2).

In the case where the condition (14) only holds,

$$\mathbb{E}\{\eta_1^2; \eta_1 \leq 1/\lambda\} \leq \mathbb{E}\{\eta_1^2; \eta_1 \leq 0\} + \mathbb{E}\{\eta_1^{1+\delta}; \eta_1 > 0\}/\lambda^{1-\delta}$$

and we deduce from the estimate (19) that

$$\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq 1/\lambda\} \leq 1 + c_6\lambda^{1+\delta}.$$

Similar to (21),

$$\mathbb{E}\{e^{\lambda\eta_1}; 1/\lambda \leq \eta_1 \leq y\} \leq c_7/x^{1+\delta'}.$$

by the condition (14). Then

$$\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq y\} \leq 1 + c_8/x^{1+\delta'} \leq e^{c_8/x^{1+\delta'}},$$

because $R(x) \leq c_9 \log x$ by (16). Hence,

$$\mathbb{P}\{T_n > x\} \leq n\overline{G}(y) + \overline{G}(x)e^{-R(x)}e^{c_8n/x^{1+\delta'}},$$

and the case of finite first moment follows. \square

Proposition 7. *Let the distribution G have mean zero, $\mathbb{E}\eta_1 = 0$, and all moments finite, $\mathbb{E}|\eta_1|^k < \infty$, $k = 1, 2, \dots$. Let $R(x)$ be the hazard function for G , that is, $\overline{G}(x) = e^{-R(x)}$. Suppose, for every $\varepsilon > 0$, there exists x_0 such that*

$$R(x)/x \leq (1 + \varepsilon)R(z)/z \quad \text{for all } x \geq z \geq x_0. \quad (22)$$

Then, for every $0 < \varepsilon < 1$, there exists a $c = c(\varepsilon) < \infty$ such that

$$\mathbb{P}\{T_n > x\} \leq (n + 1)\overline{G}(y)$$

for all $x > 0$, $y \leq (1 - \varepsilon)x$ and n such that $nR(y)/x^2 \leq 1/c$.

Proof. Take $\lambda := (1 + \varepsilon)R(y)/x$. Then $e^{-\lambda x} = e^{-(1+\varepsilon)R(y)}$.

By the condition (22),

$$\lambda z = (1 + \varepsilon)\frac{R(y)}{y}\frac{y}{x}z \leq (1 - \varepsilon^2)\frac{R(y)}{y}z \leq (1 - \varepsilon^2/2)R(z)$$

for all $z \leq y$ sufficiently large. Therefore,

$$\begin{aligned} \mathbb{E}\{e^{\lambda\eta_1}; 1/\lambda < \eta_1 \leq y\} &\leq \mathbb{E}\{e^{(1-\varepsilon^2/2)R(\eta_1)}; 1/\lambda < \eta_1\} \\ &\leq - \int_{1/\lambda}^{\infty} e^{(1-\varepsilon^2/2)R(z)} d e^{-R(z)} \\ &= \int_{1/\lambda}^{\infty} e^{-\varepsilon^2 R(z)/2} d R(z) \\ &= 2e^{-\varepsilon^2 R(1/\lambda)/2} / \varepsilon^2. \end{aligned}$$

Taking into account that, for every $\alpha > 0$, $e^{-R(x)} = o(1/x^\alpha)$ as $x \rightarrow \infty$, we get

$$\mathbb{E}\{e^{\lambda\eta_1}; 1/\lambda < \eta_1 \leq y\} = o(\lambda^2) \quad \text{as } y \rightarrow \infty. \quad (23)$$

Substituting (20) and (23) into (18) we obtain the following inequality

$$\mathbb{E}\{e^{\lambda\eta_1}; \eta_1 \leq y\} \leq 1 + cR^2(y)/x^2 \leq e^{cR^2(y)/x^2},$$

for some $c < \infty$. Hence,

$$\begin{aligned} \mathbb{P}\{T_n > x\} &\leq n\overline{G}(y) + e^{-(1+\varepsilon)R(y)}e^{c_1nR^2(y)/x^2} \\ &\leq n\overline{G}(y) + e^{-R(y)} = (n + 1)\overline{G}(y) \end{aligned}$$

in the range where $c_1nR(y)/x^2 \leq \varepsilon$ and the proof of the desired upper bound is complete. \square

In the proof above the distribution G restricted to $(-\infty, 1/\lambda]$ comes into the upper bound through its second moment only. The tail of G influences the upper bound through its values right to the point $1/\lambda$. Having this observation in mind, we formulate the following uniform version of the previous proposition for a family of distributions whose tails are ultimately dominated by that of G .

Corollary 8. *Let all the conditions of Proposition 7 be fulfilled. Let $G^{(v)}$ be a family of distributions depending on some parameter $v \in V$ such that, for some x_1 , $\overline{G^{(v)}}(x) \leq \overline{G}(x)$ for all $x > x_1$ and $v \in V$. Let every $G^{(v)}$ have mean zero and let all the second moments be bounded. Then, for every $0 < \varepsilon < 1$, there exists a $c = c(\varepsilon) < \infty$ such that*

$$\overline{(G^{(v)})^{*n}}(x) \leq (n+1)\overline{G^{(v)}}(y)$$

for all $v \in V$, $x > 0$, $y \leq (1 - \varepsilon)x$ and n such that $nR(y)/x^2 \leq 1/c$.

3 Lower bounds

Lemma 9. *Let $\mathbb{E}\xi \log \xi < \infty$. Then, for every $\varepsilon > 0$,*

$$\mathbb{P}\{W_n > x\} \geq (1 + o(1)) \sum_{i=0}^{n-1} m^i \overline{F}(m^{i+1}(1 + \varepsilon)x)$$

as $x \rightarrow \infty$ uniformly in $n \geq 1$.

Proof. Consider the following decreasing sequence of events

$$B_k(x) := \{Z_j \leq m^j x \text{ for all } j \leq k\}.$$

Since $Z_j/m^j \rightarrow W$ a.s. as $j \rightarrow \infty$,

$$\inf_{k \geq 1} \mathbb{P}\{B_k(x)\} \rightarrow 1 \quad \text{as } x \rightarrow \infty. \quad (24)$$

The events

$$A_k(x) := \{B_k(x), \xi_i^{(k)} > m^{k+1}(1 + \varepsilon)x \text{ for some } i \leq Z_k\}$$

are disjoint which implies the lower bound

$$\mathbb{P}\{W_n > x\} \geq \sum_{k=0}^{n-1} \mathbb{P}\{Z_n > m^n x \mid A_k(x)\} \mathbb{P}\{A_k(x)\}. \quad (25)$$

First we estimate the probability

$$\begin{aligned} \mathbb{P}\{A_k(x)\} &= \sum_{j=0}^{m^k x} \mathbb{P}\{B_k(x), Z_k = j\} \mathbb{P}\{\xi_i^{(k)} > m^{k+1}(1 + \varepsilon)x \text{ for some } i \leq j\} \\ &= \sum_{j=0}^{m^k x} \mathbb{P}\{B_k(x), Z_k = j\} (1 - (1 - \overline{F}(m^{k+1}(1 + \varepsilon)x))^j). \end{aligned}$$

Since $\mathbb{E}\xi \log \xi < \infty$, by the Chebyshev inequality

$$\mathbb{P}\{\xi > m^{k+1}(1+\varepsilon)x\} \leq \frac{\mathbb{E}\xi \log \xi}{m^{k+1}x \log x} = o(1/m^k x) \quad \text{as } x \rightarrow \infty \text{ uniformly in } k.$$

Hence,

$$(1 - \overline{F}(m^{k+1}(1+\varepsilon)x))^j = 1 - j\overline{F}(m^{k+1}(1+\varepsilon)x)(1 + o(1))$$

as $x \rightarrow \infty$ uniformly in $k \geq 0$ and $j \leq m^k x$. Therefore,

$$\mathbb{P}\{A_k(x)\} = (1 + o(1))\overline{F}(m^{k+1}(1+\varepsilon)x) \sum_{j=0}^{m^k x} j\mathbb{P}\{B_k(x), Z_k = j\}$$

as $x \rightarrow \infty$ uniformly in $k \geq 0$. The Kesten–Stigum theorem (see, e.g [1, Theorem 2.1]) states, in particular, that $\mathbb{E}\xi \log \xi < \infty$ if and only if the family of random variables $\{W_n, n \geq 0\}$ is uniformly integrable. Therefore, it follows from (24) that

$$\mathbb{E}\{W_k; B_k(x), W_k \leq x\} \rightarrow 1 \quad \text{as } x \rightarrow \infty \text{ uniformly in } k.$$

By this reason,

$$\begin{aligned} \sum_{j=0}^{m^k x} j\mathbb{P}\{B_k(x), Z_k = j\} &= \mathbb{E}\{Z_k; B_k(x), Z_k \leq m^k x\} \\ &= m^k \mathbb{E}\{W_k; B_k(x), W_k \leq x\} \sim m^k \end{aligned}$$

as $x \rightarrow \infty$ uniformly in $k \geq 0$. Thus, uniformly in $k \geq 0$,

$$\mathbb{P}\{A_k(x)\} = (1 + o(1))m^k \overline{F}(m^{k+1}(1+\varepsilon)x) \quad \text{as } x \rightarrow \infty. \quad (26)$$

Second we prove that

$$\inf_{n \geq 1, k \leq n-1} \mathbb{P}\{Z_n > m^n(1+\varepsilon)x \mid A_k(x)\} \rightarrow 1 \quad \text{as } x \rightarrow \infty. \quad (27)$$

Indeed, by the Markov property,

$$\begin{aligned} \mathbb{P}\{Z_n > m^n x \mid A_k(x)\} &\geq \mathbb{P}\left\{ \sum_{j=1}^{m^{k+1}(1+\varepsilon)x} Z_{n-k-1,j} > m^n x \right\} \\ &= \mathbb{P}\left\{ \sum_{j=1}^{m^{k+1}(1+\varepsilon)x} W_{n-k-1,j} > m^{k+1} x \right\}, \end{aligned}$$

where $Z_{n-k-1,j}$ are independent copies of Z_{n-k-1} and $W_{n-k-1,j}$ are independent copies of W_{n-k-1} . Since the family $\{W_n\}$ is uniformly integrable and $\mathbb{E}W_n = 1$ for every n , we may apply the law of large numbers which ensures that

$$\frac{1}{m^{k+1}x} \sum_{j=1}^{m^{k+1}(1+\varepsilon)x} W_{n-k-1,j} \xrightarrow{p} (1+\varepsilon)\mathbb{E}W_{n-k-1} = 1 + \varepsilon$$

as $x \rightarrow \infty$ uniformly in $n \geq 1$ and $k \leq n-1$. Therefore,

$$\mathbb{P}\left\{ \sum_{j=1}^{m^{k+1}(1+\varepsilon)x} W_{n-k-1,j} > m^{k+1} x \right\} \rightarrow 1,$$

which justifies the convergence (27). Substituting (26) and (27) into (25), we deduce the desired lower bound uniform in n . \square

Lemma 10. *Let the distribution F have the second moment finite, $\sigma^2 := \text{Var}\xi_1 < \infty$. Then, for every $A > 0$,*

$$\mathbb{P}\{W_n > x\} \geq \left(1 - \frac{\sigma^2}{(m^2 - m)A^2} + o(1)\right) \sum_{i=0}^{n-1} m^i \bar{F}(m^{i+1}x + A\sqrt{m^{i+1}x}) \quad (28)$$

as $x \rightarrow \infty$ uniformly in n .

In particular, if additionally the distribution F is \sqrt{x} -insensitive, then

$$\mathbb{P}\{W_n > x\} \geq (1 + o(1)) \sum_{i=0}^{n-1} m^i \bar{F}(m^{i+1}x) \quad \text{as } x \rightarrow \infty \text{ uniformly in } n. \quad (29)$$

Proof. Let events $B_k(x)$ be defined as above and

$$A_k(x) := \{B_k(x), \xi_i^{(k)} > m^{k+1}x + A\sqrt{m^{k+1}x} \text{ for some } i \leq Z_k\}$$

which again are disjoint which implies the lower bound (25). The same calculations as in the previous proof lead to the relation, uniformly in $k \geq 0$,

$$\mathbb{P}\{A_k(x)\} = (1 + o(1)) m^k \bar{F}(m^{k+1}x + A\sqrt{m^{k+1}x}) \quad \text{as } x \rightarrow \infty. \quad (30)$$

Then it remains to prove that

$$\liminf_{x \rightarrow \infty} \inf_{n \geq 1, k \leq n-1} \mathbb{P}\{Z_n > m^n x \mid A_k(x)\} \geq 1 - \frac{\sigma^2}{(m^2 - m)A^2}. \quad (31)$$

Indeed,

$$\begin{aligned} \mathbb{P}\{Z_n > m^n x \mid A_k(x)\} &\geq \mathbb{P}\left\{\sum_{j=1}^{m^{k+1}x + A\sqrt{m^{k+1}x}} Z_{n-k-1,j} > m^n x\right\} \\ &= \mathbb{P}\left\{\sum_{j=1}^{m^{k+1}x + A\sqrt{m^{k+1}x}} W_{n-k-1,j} > m^{k+1}x\right\} \\ &= \mathbb{P}\left\{\sum_{j=1}^{m^{k+1}x + A\sqrt{m^{k+1}x}} (W_{n-k-1,j} - 1) > -A\sqrt{m^{k+1}x}\right\}, \end{aligned}$$

since $\mathbb{E}W_n = 1$. Applying the Chebyshev's inequality, we deduce

$$\begin{aligned} \mathbb{P}\{Z_n > m^n x \mid A_k(x)\} &\geq 1 - \frac{\text{Var}W_{n-k-1}}{A^2} \frac{m^{k+1}x + A\sqrt{m^{k+1}x}}{m^{k+1}x} \\ &= 1 - \frac{\text{Var}W_{n-k-1}}{A^2} (1 + o(1)) \end{aligned}$$

as $x \rightarrow \infty$ uniformly in $n \geq 1$ and $k \leq n-1$. As calculated in [15, Theorem 1.5.1],

$$\text{Var}W_n = \frac{\sigma^2(1 - m^{-n})}{m^2 - m} \uparrow \frac{\sigma^2}{m^2 - m} = \text{Var}W \quad \text{as } n \rightarrow \infty,$$

which completes the proof of (31). Substituting (30) and (31) into (25) we deduce the lower bound (28).

If F is \sqrt{x} -insensitive, then letting $A \rightarrow \infty$ we conclude the second lower bound of the lemma. \square

As clearly seen from the proof of Lemma 10, in the case of Weibull distribution with parameter $\beta \in (1/2, 1)$ the tail of W_n is definitely heavier than $\overline{F}(mx)$. Now let us explain why more accurate lower bound (10) given in Introduction holds. Recalling that

$$W = \frac{1}{m} \sum_{i=1}^{\xi} W^{(i)},$$

where $W^{(i)}$ are independent copies of W which don't depend on ξ , we derive

$$\begin{aligned} \mathbb{P}\{W > x\} &\geq \mathbb{P}\left\{\sum_{i=1}^{\xi} W^{(i)} > mx; \xi \geq N_x\right\} \\ &\geq \mathbb{P}\{\xi > N_x\} \mathbb{P}\left\{\sum_{i=1}^{N_x} W^{(i)} > mx\right\}, \end{aligned} \quad (32)$$

where $N_x := [mx - z(mx)^\beta]$, $z > 0$. It is easy to see that

$$\begin{aligned} \mathbb{P}\{\xi > N_x\} &\sim e^{-(mx - z(mx)^\beta)^\beta} \\ &= e^{-(mx)^\beta + \beta z (mx)^{2\beta-1} + O(x^{3\beta-2})}. \end{aligned} \quad (33)$$

In view of log-scaled asymptotics for $\mathbb{P}\{W > x\}$ (see the first assertion of Theorem 3), $\mathbb{E}e^{(1-\varepsilon)m^\beta W^\beta} < \infty$ for every $\varepsilon > 0$. Moreover, $x^{2\beta} \ll N_x(x^\beta)^\beta$. Consequently, we may apply Nagaev's theorem [16, Theorem 3]:

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^{N_x} W^{(i)} > mx\right\} &\geq \mathbb{P}\left\{\sum_{i=1}^{N_x} (W^{(i)} - 1) > z(mx)^\beta\right\} \\ &= \exp\left\{-\frac{z^2}{2\sigma^2} (mx)^{2\beta-1} (1 + o(1))\right\}. \end{aligned} \quad (34)$$

Combining (32)–(34), we get

$$\mathbb{P}\{W > x\} \geq \exp\left\{-(mx)^\beta + (\beta z - z^2/2\sigma^2)(mx)^{2\beta-1}(1 + o(1))\right\}.$$

Maximizing $\beta z - z^2/2\sigma^2$, we obtain (10).

4 Upper bounds: a reduction to a finite time horizon

Lemma 11. *Let the distribution F be dominated varying and satisfy the condition (3). Then, for every $\varepsilon > 0$, there exists an N such that, for all $n > N$ and for all sufficiently large x ,*

$$\mathbb{P}\{W_n > x\} \leq (1 + \varepsilon)\mathbb{P}\{W_N > (1 - \varepsilon)x\}.$$

Proof. In order to derive this upper bound we write, for $z < y$,

$$\mathbb{P}\left\{\sum_{i=1}^{Z_{n-1}} \xi_i > my\right\} \leq \mathbb{P}\{Z_{n-1} > z\} + \mathbb{P}\left\{\sum_{i=1}^{Z_{n-1}} \xi_i > my; Z_{n-1} \leq z\right\}, \quad (35)$$

where the ξ 's are independent of Z_{n-1} . It follows from Proposition 6 (under the condition (14)) for sums with zero mean, $\eta_i = \xi_i - m$, that, for some $c < \infty$,

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^k \xi_i > my\right\} &= \mathbb{P}\left\{\sum_{i=1}^k (\xi_i - m) > m(y - k)\right\} \\ &\leq ck\overline{F}(m(y - k)) \quad \text{for all } k \leq z, \end{aligned}$$

provided $z \leq (y - z)^{1+\delta/2}$. Therefore,

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^{Z_{n-1}} \xi_i > my; Z_{n-1} \leq z\right\} &= \sum_{k=1}^z \mathbb{P}\{Z_{n-1} = k\} \mathbb{P}\left\{\sum_{i=1}^k \xi_i > my\right\} \\ &\leq c \sum_{k=1}^z \mathbb{P}\{Z_{n-1} = k\} k\overline{F}(m(y - k)) \quad (36) \\ &\leq c\mathbb{E}Z_{n-1}\overline{F}(m(y - z)) \\ &= cm^{n-1}\overline{F}(m(y - z)). \end{aligned}$$

Substituting this into (35) with $y = m^{n-1}x$ and $z = m^{n-1}(x - x_n)$ we obtain

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_{n-1} > x - x_n\} + cm^{n-1}\overline{F}(m^n x_n),$$

provided $x - x_n \leq m^{(n-1)\delta/2}x_n^{1+\delta/2}$. Iterating this upper bound $n - N$ times, we arrive at the following inequality:

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_N > x - x_n - \dots - x_{N+1}\} + c \sum_{k=N+1}^n m^{k-1}\overline{F}(m^k x_k), \quad (37)$$

provided $x \leq m^{(k-1)\delta/2}x_k^{1+\delta/2}$ for all k . Take decreasing sequence $x_k = x/k^2$. Choose N so large that $m^{(k-1)\delta/2} \geq k^{2+\delta}$ for all $k \geq N+1$. Then (37) holds for all $n \geq N+1$ and we have

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_N > (1 - 1/(N+1)^2 - \dots - 1/n^2)x\} + c \sum_{k=N+1}^n m^{k-1}\overline{F}(m^k x/k^2).$$

Choose N so large that additionally $\sum_{k=N}^{\infty} 1/k^2 \leq \varepsilon$. Then

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_N > (1 - \varepsilon)x\} + c \sum_{k=N+1}^n m^{k-1}\overline{F}(m^k x/k^2).$$

Owing to the condition (3),

$$\sum_{k=N+1}^n m^{k-1}\overline{F}(m^k x/k^2) \leq c_1\overline{F}(mx) \sum_{k=N+1}^{\infty} \frac{m^{k-1}}{(m^{k-1}/k^2)^{1+\delta}}.$$

Now we may increase N so that

$$c \sum_{k=N+1}^n m^{k-1}\overline{F}(m^k x/k^2) \leq \varepsilon\overline{F}(mx)/3,$$

which implies

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_N > (1 - \varepsilon)x\} + \varepsilon \bar{F}(mx)/2.$$

Applying here Lemma 9, we deduce $\bar{F}(mx) \leq (1 + o(1))\mathbb{P}\{W_N > (1 - \varepsilon)x\}$ as $x \rightarrow \infty$, so

$$\mathbb{P}\{W_n > x\} \leq (1 + \varepsilon)\mathbb{P}\{W_N > (1 - \varepsilon)x\}$$

for all sufficiently large x , and the proof is complete. \square

The calculations above imply the following

Corollary 12. *Let the distribution F be dominated varying and satisfy the condition (3). Then there exists a constant $c < \infty$ such that $\mathbb{P}\{W_n > x\} \leq c\bar{F}(x)$ for all n and x .*

For dominated varying distributions it is possible to obtain more accurate bound which will be of use for wider class of distributions than intermediate regularly varying. We do it in the next lemma where the bound provided by the previous corollary serves as the first step preliminary bound.

Lemma 13. *Let $\mathbb{E}\xi^2 < \infty$, the distribution F be dominated varying and satisfy the condition (3). Then, for every $\gamma > 1/2$ and $\varepsilon > 0$, there exists an N such that, for all $n > N$ and for all sufficiently large x ,*

$$\mathbb{P}\{W_n > x\} \leq (1 + \varepsilon)\mathbb{P}\{W_N > x - x^\gamma\}.$$

Proof. Here we need more accurate upper bounds based on (36). Take $\delta \in (1/\gamma - 1, 1)$. First note that, as follows from Proposition 6 under the condition (14) (which is fulfilled because $\mathbb{E}\xi^2 < \infty$), the bound (36) now holds within a larger time range where $z \leq (y - z)^{1+\delta}$. For those z ,

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^{Z_{n-1}} \xi_i > my; Z_{n-1} \leq z\right\} &\leq c\left(\sum_{k=1}^{z/2} + \sum_{k=z/2}^z\right)\mathbb{P}\{Z_{n-1} = k\}k\bar{F}(m(y - k)) \\ &=: c(\Sigma_1 + \Sigma_2). \end{aligned}$$

We have

$$\begin{aligned} \Sigma_1 &\leq \bar{F}(m(y - z/2)) \sum_{k=1}^{y/2} \mathbb{P}\{Z_{n-1} = k\}k \\ &\leq \mathbb{E}Z_{n-1}\bar{F}(my/2) \leq c_1 m^{n-1}\bar{F}(my), \end{aligned}$$

for some $c_1 < \infty$, by dominated variation of F . Further,

$$\begin{aligned} \Sigma_2 &\leq \mathbb{P}\{Z_{n-1} > z/2\}z\bar{F}(m(y - z)) \\ &\leq c_2 \bar{F}(z/2m^{n-1})z\bar{F}(m(y - z)) \\ &\leq c_2 c_1 \bar{F}(z/m^{n-1})z\bar{F}(m(y - z)), \end{aligned}$$

by Corollary 12 and dominated variation of F . Collecting bounds for Σ_1 and Σ_2 with $y = m^{n-1}x$ and $z = m^{n-1}(x - x_n)$, we obtain from (35) that

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_{n-1} > x - x_n\} + c_1 m^{n-1} \bar{F}(m^n x) + c_3 \bar{F}(x - x_n) m^{n-1} x \bar{F}(m^n x_n)$$

provided $x - x_n \leq m^{(n-1)\delta} x_n^{1+\delta}$. Iterating this upper bound $n - N$ times, we arrive at the following inequality:

$$\begin{aligned} \mathbb{P}\{W_n > x\} &\leq \mathbb{P}\{W_N > x - x_n - \dots - x_{N+1}\} + c_1 \sum_{k=N+1}^n m^{k-1} \bar{F}(m^k(x - x_n - \dots - x_{k+1})) \\ &\quad + c_3 \sum_{k=N+1}^n \bar{F}(x - x_n - \dots - x_k) m^{k-1} x \bar{F}(m^k x_k), \end{aligned} \quad (38)$$

provided $x \leq m^{(k-1)\delta} x_k^{1+\delta}$ for all $k = n, \dots, N+1$.

Now take decreasing sequence $x_k = x^\gamma/k^2$. Since $\gamma > 1/2$ and $\delta \in (1/\gamma - 1, 1)$, $x^{\gamma(1+\delta)} > x$. Then (38) holds for every $n \geq N+1$ and we have

$$\begin{aligned} \mathbb{P}\{W_n > x\} &\leq \mathbb{P}\{W_N > x - (1/(N+1)^2 + \dots + 1/n^2)x^\gamma\} \\ &\quad + c_1 \sum_{k=N+1}^n m^{k-1} \bar{F}(m^k(x - (1/n^2 + \dots + 1/(k+1)^2)x^\gamma)) \\ &\quad + c_3 \sum_{k=N+1}^n \bar{F}(x - (1/n^2 + \dots + 1/k^2)x^\gamma) m^{k-1} x \bar{F}(m^k x^\gamma/k^2). \end{aligned}$$

Choose N so large that $\sum_{k=N+1}^\infty 1/k^2 \leq 1$. Then

$$\begin{aligned} \mathbb{P}\{W_n > x\} &\leq \mathbb{P}\{W_N > x - x^\gamma\} + c_1 \bar{F}(x - x^\gamma) \sum_{k=N+1}^n m^{k-1} \frac{\bar{F}(m^k(x - x^\gamma))}{\bar{F}(x - x^\gamma)} \\ &\quad + c_3 \bar{F}(x - x^\gamma) \sum_{k=N+1}^n m^{k-1} x \bar{F}(m^k x^\gamma/k^2). \end{aligned}$$

Owing to the condition (3),

$$\sum_{k=N+1}^n m^{k-1} \frac{\bar{F}(m^k(x - x^\gamma))}{\bar{F}(x - x^\gamma)} \leq c_4 \sum_{k=N+1}^\infty \frac{m^{k-1}}{m^{k(1+\delta)}} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

and

$$\begin{aligned} \sum_{k=N+1}^n m^{k-1} x \bar{F}(m^k x^\gamma/k^2) &\leq c_4 x \bar{F}(x^\gamma) \sum_{k=N+1}^n \frac{m^{k-1}}{(m^k/k^2)^{1+\delta}} \\ &\leq c_4 \mathbb{E} \xi^2 x^{1-2\gamma} \sum_{k=N+1}^\infty \frac{m^{k-1}}{(m^k/k^2)^{1+\delta}} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Taking into account that $\bar{F}(x - x^\gamma) \leq c_5 \bar{F}(mx)$ and further increasing N we derive the following bound:

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_N > x - x^\gamma\} + \varepsilon \bar{F}(mx)/2.$$

Applying here Lemma 9, we deduce $\overline{F}(mx) \leq (1 + o(1))\mathbb{P}\{W_N > x\}$ as $x \rightarrow \infty$, so

$$\mathbb{P}\{W_n > x\} \leq (1 + \varepsilon)\mathbb{P}\{W_N > x - x^\gamma\}$$

for all sufficiently large x , and the proof is complete. \square

Note that the assertion of Lemma 11 holds not only for intermediate regularly varying distributions but for Weibull distributions as well; more precisely, the following result holds.

Lemma 14. *Let $\overline{F}(x) = e^{-R(x)}$ where $R(x)$ satisfies the condition (22) and $R(x)/x \rightarrow 0$. Let the condition (3) hold. Then, for every $\varepsilon > 0$, there exists an N such that*

$$\mathbb{P}\{W_n > x\} \leq (1 + \varepsilon)\mathbb{P}\{W_N > (1 - \varepsilon)x\}$$

for all $n > N$ and for all sufficiently large x .

Proof is similar to that of Lemma 11. Start again with the inequality (35). As follows from Proposition 7 for sums with zero mean, $\eta_i = \xi_i - m$, that, for some $c < \infty$,

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^k \xi_i > my\right\} &= \mathbb{P}\left\{\sum_{i=1}^k (\xi_i - m) > m(y - k)\right\} \\ &\leq ck\overline{F}((m - \varepsilon/2)(y - k)) \quad \text{for all } k \leq z, \end{aligned}$$

provided $z \leq \frac{(y-z)^2}{cR(y-z)}$. Therefore,

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^{Z_{n-1}} \xi_i > my; Z_{n-1} \leq z\right\} &\leq c\mathbb{E}Z_{n-1}\overline{F}((m - \varepsilon/2)(y - z)) \\ &= cm^{n-1}\overline{F}((m - \varepsilon/2)(y - z)). \end{aligned}$$

Substituting this into (35) with $y = m^{n-1}x$ and $z = m^{n-1}(x - x_n)$ we obtain

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_{n-1} > x - x_n\} + cm^{n-1}\overline{F}((m - \varepsilon/2)^n x_n),$$

provided $x - x_n \leq \frac{m^{n-1}x_n^2}{cR(m^{n-1}x_n)}$. Iterating this upper bound $n - N$ times, we arrive at the following inequality:

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_N > x - x_n - \dots - x_{N+1}\} + c \sum_{k=N+1}^n m^{k-1}\overline{F}((m - \varepsilon/2)^k x_k) \quad (39)$$

provided $x - x_k \leq \frac{m^{k-1}x_k^2}{cR(m^{k-1}x_k)}$ for all $k = n, \dots, N + 1$. Take decreasing sequence $x_k = x/k^2$. Choose N so large that $\frac{m^{k-1}x}{k^2} \geq R(m^{k-1}x/k^2)$ for all $k \geq N + 1$; it is possible because $R(z)/z \rightarrow 0$ as $z \rightarrow \infty$. Then (39) holds for every $n \geq N + 1$ and we have

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\{W_N > (1 - 1/(N + 1)^2 - \dots - 1/n^2)x\} + c \sum_{k=N+1}^n m^{k-1}\overline{F}((m - \varepsilon/2)^k x/k^2).$$

Choose $\varepsilon > 0$ so small to satisfy $m < (m - \varepsilon/2)^{1+\delta}$ where $\delta > 0$ is taken from the condition (3). Then the rest of the proof is the same as the proof of Lemma 11. \square

5 Finite time horizon asymptotics

As follows from [10, Section 6] for intermediate regularly varying distribution F , for every fixed n ,

$$\mathbb{P}\{W_n > x\} \sim \sum_{i=0}^{n-1} m^i \bar{F}(m^{i+1}x) \quad \text{as } x \rightarrow \infty. \quad (40)$$

For the case where the second moment of ξ is finite, we extend this result for a wider class of distributions as follows.

Lemma 15. *Let $\mathbb{E}\xi^2 < \infty$ and the distribution F be dominated varying. If F is x^γ -insensitive for some $\gamma > 1/2$, then the equivalence (40) holds for every fixed n .*

Proof. First, Lemma 10 guarantees the right lower bound. The upper bound will be proved by induction. It is true for $n = 1$. Assume, for some n ,

$$\mathbb{P}\{W_n > x\} \leq (1 + o(1)) \sum_{i=0}^{n-1} m^i \bar{F}(m^{i+1}x) \quad \text{as } x \rightarrow \infty. \quad (41)$$

Prove that then (41) holds for $n + 1$. Start with the inequality

$$\begin{aligned} \mathbb{P}\{W_{n+1} > x\} &= \mathbb{P}\left\{\sum_{i=1}^{Z_n} \xi_i > m^{n+1}x\right\} \\ &\leq \mathbb{P}\{Z_n > m^n(x - x^\gamma)\} + \mathbb{P}\left\{\sum_{i=1}^{Z_n} \xi_i > m^{n+1}x; m^n x/2 < Z_n \leq m^n(x - x^\gamma)\right\} \\ &\quad + \mathbb{P}\left\{\sum_{i=1}^{Z_n} \xi_i > m^{n+1}x; m^n x^\gamma < Z_n \leq m^n x/2\right\} \\ &\quad + \mathbb{P}\left\{\sum_{i=1}^{Z_n} \xi_i > m^{n+1}x; Z_n \leq m^n x^\gamma\right\} \\ &=: P_1 + P_2 + P_3 + P_4, \end{aligned}$$

where the ξ 's are independent of Z_n . Due to the induction hypothesis and since F is x^γ -insensitive,

$$\begin{aligned} P_1 &= \mathbb{P}\{W_n > x - x^\gamma\} \\ &\leq (1 + o(1)) \sum_{i=0}^{n-1} m^i \bar{F}(m^{i+1}(x - x^\gamma)) \\ &\sim \sum_{i=0}^{n-1} m^i \bar{F}(m^{i+1}x) \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Take $\delta \in (1/\gamma - 1, 1)$. All the values of k not greater than $m^n x$ are negligible compared to $y^{1+\delta}$ where $y = m^{n+1}x^\gamma$, $\gamma > 1/2$. Therefore, by Proposition 6 there exists $c < \infty$ such that

$$\mathbb{P}\left\{\sum_{i=1}^k (\xi_i - m) > m^{n+1}x - km\right\} \leq ck \bar{F}(m^{n+1}x - km)$$

for sufficiently large x and for all $k \leq m^n(x - x^\gamma)$.

Therefore, for sufficiently large x ,

$$\begin{aligned}
P_2 &= \sum_{k=m^n x/2}^{m^n(x-x^\gamma)} \mathbb{P}\{Z_n = k\} \mathbb{P}\left\{\sum_{i=1}^k (\xi_i - m) > m^{n+1}x - km\right\} \\
&\leq c \sum_{k=m^n x/2}^{m^n(x-x^\gamma)} \mathbb{P}\{Z_n = k\} k \bar{F}(m^{n+1}x - km) \\
&\leq cm^n x \mathbb{P}\{Z_n \geq m^n x/2\} \bar{F}(m^{n+1}x^\gamma).
\end{aligned}$$

Since $\mathbb{E}\xi^2 < \infty$ and $\gamma > 1/2$, $x\bar{F}(m^{n+1}x^\gamma) \rightarrow 0$ as $x \rightarrow \infty$. Hence, as $x \rightarrow \infty$,

$$\begin{aligned}
P_2 &= o(\mathbb{P}\{W_n \geq x/2\}) \\
&= o(\bar{F}(mx/2)) = o(\bar{F}(mx)),
\end{aligned}$$

owing the induction hypothesis (41) and dominated variation of F .

Further, for sufficiently large x ,

$$\begin{aligned}
P_3 &\leq c \sum_{k=m^n x^\gamma}^{m^n x/2} \mathbb{P}\{Z_n = k\} k \bar{F}(m^{n+1}x - km) \\
&\leq c \mathbb{E}\{Z_n; Z_n > m^n x^\gamma\} \bar{F}(m^{n+1}x/2) \\
&= o(\bar{F}(mx)) \quad \text{as } x \rightarrow \infty,
\end{aligned}$$

again because of dominated variation of F .

Finally,

$$P_4 = \sum_{k=1}^{m^n x^\gamma} \mathbb{P}\{Z_n = k\} \mathbb{P}\left\{\sum_{i=1}^k (\xi_i - 2m) > m^{n+1}x - 2km\right\}.$$

The distribution F is dominated varying and long-tailed (constant-insensitive) which implies it belongs to the class \mathcal{S}^* , see, e.g. [13, Theorem 3.29]. Also, the expression $m^{n+1}x - 2km$ tends to infinity as $x \rightarrow \infty$ uniformly in $k \leq mx^\gamma$. This allows to apply here Proposition 5 for random variables $\eta_i := \xi_i - 2m$ with negative mean; it ensures that, uniformly in $k \leq mx^\gamma$,

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^k (\xi_i - 2m) > m^{n+1}x - 2km\right\} &\leq (1 + o(1)) k \bar{F}(m^{n+1}x - 2km) \\
&\sim k \bar{F}(m^{n+1}x) \quad \text{as } x \rightarrow \infty,
\end{aligned}$$

because F is x^γ -insensitive. Thus,

$$\begin{aligned}
P_4 &\sim \bar{F}(m^{n+1}x) \sum_{k=1}^{m^n x^\gamma} \mathbb{P}\{Z_n = k\} k \\
&\sim \bar{F}(m^{n+1}x) \mathbb{E}Z_n = m^n \bar{F}(m^{n+1}x).
\end{aligned}$$

Combining bounds for P_1 through P_4 we deduce that

$$\mathbb{P}\{W_{n+1} > x\} \leq \mathbb{P}\{W_n > mx\} + m^n \bar{F}(m^{n+1}x) + o(\bar{F}(mx)) \quad \text{as } x \rightarrow \infty,$$

and the induction hypothesis (41) completes the proof. \square

If the distribution F is rapidly varying then

$$\sum_{i=0}^{\infty} m^i \overline{F}(m^{i+1}x) \sim \overline{F}(mx) \quad \text{as } x \rightarrow \infty. \quad (42)$$

Indeed, fix $\varepsilon > 0$ and choose $x(\varepsilon)$ such that $\overline{F}(mx) \leq \varepsilon \overline{F}(x)$ for every $x > x(\varepsilon)$. Then, for $x > x(\varepsilon)$,

$$\sum_{i=1}^{\infty} m^i \overline{F}(m^{i+1}x) \leq \sum_{i=1}^{\infty} (m\varepsilon)^i \overline{F}(mx) = \frac{m\varepsilon}{1 - m\varepsilon} \overline{F}(mx).$$

The constant multiplier on the right side may be made as small as we please by appropriate choice of ε , so the equivalence (42) follows.

Lemma 16. *Let $\overline{F}(x) = e^{-R(x)}$ where $R(x)$ is regularly varying with index $\beta \in (0, 1/2)$. In the case $\beta \in [\frac{3-\sqrt{5}}{2}, 1/2)$ assume also that the condition (8) holds. Additionally assume that $F \in \mathcal{S}^*$. Then, for every fixed n ,*

$$\mathbb{P}\{W_n > x\} \sim \overline{F}(mx) \quad \text{as } x \rightarrow \infty.$$

Proof. Since $\beta < 1/2$, the distribution F is \sqrt{x} -insensitive which by Lemma 10 implies the lower bound $\mathbb{P}\{W_n > x\} \geq (1 + o(1))\overline{F}(mx)$ as $x \rightarrow \infty$.

To prove the upper bound, apply induction arguments. For $n = 1$, we have the equality $\mathbb{P}\{W_1 > x\} = \overline{F}(mx)$. Assume now $\mathbb{P}\{W_n > x\} \sim \overline{F}(mx)$ for some $n \geq 1$. Prove that then it holds for $n + 1$.

If $\beta < \frac{3-\sqrt{5}}{2}$ then the interval $(\frac{1}{2-\beta}, 1 - \beta)$ is not empty; in this case we take $\gamma_1 = \gamma_2 \in (\frac{1}{2-\beta}, 1 - \beta)$. If $\beta \in [\frac{3-\sqrt{5}}{2}, 1/2)$ then $\frac{1}{2-\beta} \geq 1 - \beta$ and we take $\gamma_1 \in (1/2, 1 - \beta)$ and $\gamma_2 > 1/(2 - \beta)$ so that $\gamma_2 \geq \gamma_1$. Since $\gamma_1 < 1 - \beta$, F is x^{γ_1} -insensitive. Start with the inequality

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^{Z_n} \xi_i > m^{n+1}x\right\} &\leq \mathbb{P}\{Z_n > m^n(x - x^{\gamma_1})\} + \mathbb{P}\left\{\sum_{i=1}^{Z_n} \xi_i > m^{n+1}x; Z_n \leq m^n(x - x^{\gamma_1})\right\} \\ &=: P_1 + P_2, \end{aligned}$$

where the ξ 's do not depend on Z_n . By the induction hypothesis and since F is x^{γ_1} -insensitive,

$$P_1 \sim \overline{F}(m(x - x^{\gamma_1})) \sim \overline{F}(mx) \quad \text{as } x \rightarrow \infty.$$

It remains to prove that $P_2 = o(\overline{F}(mx))$ as $x \rightarrow \infty$. Start with the following decomposition:

$$\begin{aligned} P_2 &= \left(\sum_{k=1}^{m^n(x-x^{\gamma_2})-1} + \sum_{k=m^n(x-x^{\gamma_2})}^{m^n(x-x^{\gamma_1})} \right) \mathbb{P}\{Z_n = k\} \mathbb{P}\left\{\sum_{i=1}^k (\xi_i - m) > m^{n+1}x - km\right\} \\ &=: P_{21} + P_{22}. \end{aligned}$$

In the first sum P_{21} we have $m^{n+1}x - km \geq m^{n+1}x^{\gamma_2} \gg x^{1/(2-\beta)}$ due to the choice $\gamma_2 > 1/(2 - \beta)$. The function $R(x)/x^2$ is regularly varying with index $\beta - 2$. Hence $kR(m^{n+1}x - km)/(m^{n+1}x - km)^2 \rightarrow 0$ as $x \rightarrow \infty$ uniformly in $k \leq m^n(x - x^{\gamma_2})$. This

observation together with regular variation of $R(x)$ allows us to apply Proposition 7 with $y = (1 - \varepsilon)x$ which ensures that

$$\mathbb{P}\left\{\sum_{i=1}^k (\xi_i - m) > m^{n+1}x - mk\right\} \leq k\bar{F}((m^{n+1}x - mk)(1 - \varepsilon))$$

for sufficiently large x and for all $k \leq m^n(x - x^{\gamma_2})$. Thus, for sufficiently large x ,

$$P_{21} \leq \sum_{k=1}^{m^n(x-x^{\gamma_2})} \mathbb{P}\{Z_n = k\}k\bar{F}((m^n x - k)m(1 - \varepsilon)).$$

Take $\varepsilon > 0$ so small that $m(1 - \varepsilon) > 1$. Then by rapid variation of F , as $x \rightarrow \infty$,

$$\bar{F}((m^n x - k)m(1 - \varepsilon)) = o(\bar{F}(m^n x - k)) \quad \text{uniformly in } k \leq m^n(x - x^{\gamma_2}).$$

In addition, owing the induction hypothesis,

$$\mathbb{P}\{Z_n = k\} \leq \mathbb{P}\{W_n \geq k/m^n\} \leq c\bar{F}(k/m^{n-1}),$$

for some $c < \infty$. Thus, as $x \rightarrow \infty$,

$$\begin{aligned} P_{21} &\leq o(1) \int_0^{m^n(x-x^{\gamma_2})} y\bar{F}(y/m^{n-1})\bar{F}(m^n x - y)dy \\ &= o(1) \int_0^{m(x-x^{\gamma_2})} y\bar{F}(y)\bar{F}(m^{n-1}(mx - y))dy. \end{aligned}$$

Since $m^{n-1} \geq m > 1$ and $\beta > 0$,

$$y\bar{F}(m^{n-1}(mx - y)) = o(\bar{F}(mx - y))$$

as $x \rightarrow \infty$ uniformly in $y \leq m(x - x^{\gamma_2})$. Therefore,

$$P_{21} \leq o(1) \int_0^{mx} \bar{F}(y)\bar{F}(mx - y)dy.$$

The inclusion $F \in \mathcal{S}^*$ means that

$$\int_0^{mx} \bar{F}(y)\bar{F}(mx - y)dy \sim 2\bar{F}(mx) \int_0^\infty \bar{F}(y)dy \quad \text{as } x \rightarrow \infty,$$

which finally implies $P_{21} = o(\bar{F}(mx))$. In the case $\beta < \frac{3-\sqrt{5}}{2}$ this completes the proof because then $\gamma_1 = \gamma_2$ and $P_{22} = 0$.

If $\beta < 1/2$ then it remains to prove that $P_{22} = o(\bar{F}(mx))$ too. We have

$$P_{22} = \sum_{k=m^n x^{\gamma_1}}^{m^n x^{\gamma_2}} \mathbb{P}\{Z_n = m^n x - k\} \mathbb{P}\left\{\sum_{i=1}^{m^n x - k} (\xi_i - m) > mk\right\}.$$

By the induction hypothesis

$$\mathbb{P}\{Z_n = m^n x - k\} \leq \mathbb{P}\{W_n \geq x - k/m^n\} \sim \bar{F}(mx - k/m^{n-1}),$$

so that

$$P_{22} \leq c_1 \sum_{k=m^n x^{\gamma_1}}^{m^n x^{\gamma_2}} \bar{F}(mx - k/m^{n-1})(m^n x - k + 1) \bar{F}(y_k)$$

for any y_k satisfying the inequalities $y_k \leq mk/2$ and $m^n x - k \leq (mk)^2/cR(y_k)$, where $c = c(1/2)$ is defined in Proposition 7. Choose $\gamma \in (2\beta, 1)$ such that

$$\frac{1}{\gamma_1} - 1 < \gamma < \frac{1}{\gamma_2} - 1 + \beta, \quad (43)$$

it is possible if we choose $\gamma_2 > 1/(2 - \beta)$ sufficiently close to $1/(2 - \beta)$. Then take y_k which solves $R(y_k) = m^{2-n}k^{1+\gamma}/cx = c_2k^{1+\gamma}/x$. With this choice, $y_k \leq mk/2$ for $k \leq m^n x^{\gamma_2}$ and sufficiently large x , by the right inequality in (43), and $m^n x - k \leq (mk)^2/cR(y_k)$.

Further, since $\bar{F}(y_k) = e^{-R(y_k)}$,

$$\begin{aligned} P_{22} &\leq c_3 x \sum_{k=m^n x^{\gamma_1}}^{m^n x^{\gamma_2}} \bar{F}(mx - k/m^{n-1}) \bar{F}(y_k) \\ &\leq c_3 x \bar{F}(mx) \sum_{k=m^n x^{\gamma_1}}^{m^n x^{\gamma_2}} e^{R(mx) - R(mx - k/m^{n-1}) - R(y_k)}. \end{aligned}$$

By the condition (8) on the increments of R and by regular variation of R we have

$$\frac{R(x) - R(y)}{x - y} \leq c_4 \frac{R(x)}{x}, \quad x \geq y \geq 1,$$

which implies

$$\begin{aligned} R(mx) - R(mx - k/m^{n-1}) - R(y_k) &\leq c_5 k R(mx)/x - c_2 k^{1+\gamma}/x \\ &= (c_5 R(mx) - c_2 k^\gamma)k/x. \end{aligned}$$

Since $R(mx)$ is regularly varying with index $\beta < 1/2$ and $k \geq m^n x^{\gamma_1}$, the choice $\gamma_1 \in (1/2, 1 - \beta)$ and $\gamma \in (2\beta, 1)$ ensures $R(mx) = o(k)$. Hence,

$$R(mx) - R(mx - k/m^{n-1}) - R(y_k) \leq -c_6 k^{1+\gamma}/x,$$

which yields

$$P_{22} \leq c_4 x \bar{F}(mx) \sum_{k=m^n x^{\gamma_1}}^{\infty} e^{-c_6 k^{1+\gamma}/x} = o(\bar{F}(mx)) \quad \text{as } x \rightarrow \infty,$$

due to $\gamma_1(1 + \gamma) > 1$, by the left inequality in (43). Combining altogether we deduce that $P_2 = o(\bar{F}(mx))$ and consequently $\mathbb{P}\{W_{n+1} > x\} \sim P_1 \sim \bar{F}(mx)$ as $x \rightarrow \infty$ and the proof is complete. \square

6 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1. The bounds (4) follow from Lemma 9 and Corollary 12. All other assertions follow from the equivalence (40) and from Lemmas 9 and 11. \square

Proof of Theorem 2 follows from Lemmas 15, 10 and 13. \square

Proof of Theorem 3. The lower bound for the general case $\beta < 1$ follows from Lemma 9. The upper bound follows from Lemma 14 which reduces the problem to the finite time horizon N and further induction arguments like

$$\begin{aligned} \mathbb{P}\{W_N > x\} &= \mathbb{P}\left\{\sum_{i=1}^{\xi} W_{N-1}^{(i)} > mx\right\} \\ &\leq \mathbb{P}\{\xi > mx(1-\varepsilon)\} + \mathbb{P}\left\{\sum_{i=1}^{\xi} W_{N-1}^{(i)} > mx; \xi \leq mx(1-\varepsilon)\right\}, \end{aligned}$$

where $W_{N-1}^{(1)}, W_{N-1}^{(2)}, \dots$ are independent copies of W_{N-1} . Assuming that W_{N-1} has a tail not heavier than $c\bar{F}((1-\varepsilon)x)$ we may estimate here the second probability as follows:

$$\mathbb{P}\left\{\sum_{i=1}^{\xi} W_{N-1}^{(i)} > mx; \xi \leq mx(1-\varepsilon)\right\} = \sum_{k=1}^{mx(1-\varepsilon)} \mathbb{P}\{\xi = k\} \mathbb{P}\left\{\sum_{i=1}^k (W_{N-1}^{(i)} - k) > mx - k\right\}.$$

By Proposition 7,

$$\mathbb{P}\left\{\sum_{i=1}^k (W_{N-1}^{(i)} - k) > mx - k\right\} \leq (k+1)\bar{F}((1-\varepsilon)(mx-k))$$

as $x \rightarrow \infty$ uniformly in $k \leq mx(1-\varepsilon)$; note that the condition $k \leq mx(1-\varepsilon)$ implies $mx - k \geq mx\varepsilon$ and hence covers both conditions of Proposition 7. Thus,

$$\begin{aligned} \sum_{k=1}^{mx(1-\varepsilon)} \mathbb{P}\{\xi = k\} \mathbb{P}\left\{\sum_{i=1}^k (W_{N-1}^{(i)} - k) > mx - k\right\} &\leq 2 \sum_{k=1}^{mx(1-\varepsilon)} \mathbb{P}\{\xi = k\} k\bar{F}((1-\varepsilon)(mx-k)) \\ &= o(\bar{F}(m(1-2\varepsilon)x)) \end{aligned}$$

as $x \rightarrow \infty$, by standard properties of Weibull type distributions. This completes the proof of upper bound for the case $\beta < 1$.

In the case $\beta < 1/2$ the distribution F is \sqrt{x} -insensitive which by Lemma 10 implies the lower bound $\mathbb{P}\{W_n > x\} \geq (1+o(1))\bar{F}(mx)$ as $x \rightarrow \infty$.

Now prove the upper bound for the case $\beta < 1/2$. Fix $\varepsilon > 0$. Owing Lemma 14 we find N so that, for all $n > N$ and for all sufficiently large x ,

$$\mathbb{P}\{W_n > x\} \leq (1+\varepsilon)\mathbb{P}\{W_N > (1-\varepsilon)x\}.$$

As in the proof of Lemma 16, take $\gamma \in (1/(2-\beta), 1-\beta)$ so F is x^γ -insensitive. Make use of the decomposition, for $n > N+1$,

$$\begin{aligned} \mathbb{P}\{W_n > x\} &\leq \mathbb{P}\{\xi > m(x-x^\gamma)\} + \mathbb{P}\left\{\sum_{i=1}^{\xi} W_{n-1}^{(i)} > mx; \xi \leq m(x-x^\gamma)\right\} \\ &=: P_1 + P_2. \end{aligned}$$

Since F is x^γ -insensitive,

$$P_1 = \mathbb{P}\{\xi > m(x - x^\gamma)\} \sim \overline{F}(mx) \quad \text{as } x \rightarrow \infty.$$

Further, make use of Lemma 14 which is applicable because $n - 1 > N$: ultimately in y ,

$$\begin{aligned} \mathbb{P}\{W_{n-1} > y\} &\leq (1 + \varepsilon)\mathbb{P}\{W_N > (1 - \varepsilon)y\} \\ &\leq (1 + 2\varepsilon)\overline{F}((1 - \varepsilon)my), \end{aligned}$$

by virtue of Lemma 16. Choose $\varepsilon > 0$ so small that $m_* := (1 - \varepsilon)m > 1$, it is possible due to $m > 1$. The family $\{W_{n-1} - 1, n > N + 1\}$ satisfies the conditions of Corollary 8 which further allows to prove that $P_2 = o(\overline{F}(mx))$ as $x \rightarrow \infty$ uniformly in $n > N + 1$ in the same way as in the proof of Lemma 16. \square

7 The case of regularly varying tail with index -1 ; proof of Theorem 4

As proven in Lemma 9, for every $\varepsilon > 0$,

$$\mathbb{P}\{W_n > x\} \geq (1 + o(1)) \sum_{k=0}^{n-1} m^k \overline{F}(m^{k+1}(1 + \varepsilon)x)$$

as $x \rightarrow \infty$ uniformly in $n \geq 1$. Since F is regularly varying, we deduce from here that

$$\mathbb{P}\{W_n > x\} \geq (1 + o(1)) \sum_{k=0}^{n-1} m^k \overline{F}(m^{k+1}x)$$

as $x \rightarrow \infty$ uniformly in $n \geq 1$. Then it remains to prove the following upper bound: for every fixed $\varepsilon > 0$,

$$\mathbb{P}\{W_n > x\} \leq (1 + o(1)) \sum_{k=0}^{n-1} m^k \overline{F}(m^{k+1}x(1 - \varepsilon)). \quad (44)$$

The method for proving upper bounds based on Lemma 11 doesn't work here because it essentially requires the condition (3). By this reason we proceed in a different way. Define events

$$A_k(x) := \{\xi_i^{(k)} > m^{k+1}x(1 - \varepsilon) \text{ for some } i \leq Z_k\}.$$

Clearly,

$$\mathbb{P}\{A_k(x) | Z_k = j\} \leq j \overline{F}(m^{k+1}x(1 - \varepsilon)), \quad j \geq 1.$$

Therefore, $\mathbb{P}\{A_k(x)\} \leq m^k \overline{F}(m^{k+1}x(1 - \varepsilon))$ and

$$\mathbb{P}\left\{\bigcup_{k=0}^{n-1} A_k(x)\right\} \leq \sum_{k=0}^{n-1} \mathbb{P}\{A_k(x)\} \leq \sum_{k=0}^{n-1} m^k \overline{F}(m^{k+1}x(1 - \varepsilon)).$$

Owing to this and the upper bound

$$\mathbb{P}\{W_n > x\} \leq \mathbb{P}\left\{W_n > x, \bigcap_{k=0}^{n-1} \overline{A_k(x)}\right\} + \mathbb{P}\left\{\bigcup_{k=0}^{n-1} A_k(x)\right\},$$

we conclude that (44) will be implied by the following relation: for every fixed $\varepsilon > 0$,

$$\mathbb{P}\left\{W_n > x, \bigcap_{k=0}^{n-1} \overline{A_k(x)}\right\} = o\left(\sum_{k=0}^{n-1} m^k \overline{F}(m^{k+1}x)\right) \quad (45)$$

as $x \rightarrow \infty$ uniformly in $n \geq 1$. By the Chebyshev inequality, for every $\lambda > 0$

$$\begin{aligned} \mathbb{P}\left\{W_n > x, \bigcap_{k=0}^{n-1} \overline{A_k(x)}\right\} &\leq \frac{\mathbb{E}\{e^{\lambda Z_n} - 1; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\}}{e^{\lambda m^n x} - 1} \\ &= \frac{\mathbb{E}\{e^{\lambda Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\} - 1}{e^{\lambda m^n x} - 1} + \frac{\mathbb{P}\{\bigcup_{k=0}^{n-1} A_k(x)\}}{e^{\lambda m^n x} - 1}, \end{aligned}$$

so that the relation (45) will follow if we find $\lambda = \lambda_n(x)$ such that

$$\lambda m^n x \rightarrow \infty \quad (46)$$

and

$$\frac{\mathbb{E}\{e^{\lambda Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\} - 1}{e^{\lambda m^n x} - 1} = o\left(\sum_{k=0}^{n-1} m^k \overline{F}(m^{k+1}x)\right). \quad (47)$$

In order to find $\lambda = \lambda_n(x)$ satisfying (46) and (47) we proceed with a suitable exponential bounds for bounded random variables. Take $\lambda_{nn} > 0$ and consider the following exponential moment

$$\begin{aligned} \mathbb{E}\left\{e^{\lambda_{nn} Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\right\} &= \sum_{i=1}^{\infty} \mathbb{E}\left\{e^{\lambda_{nn} Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}, Z_{n-1} = i\right\} \\ &= \sum_{i=1}^{\infty} \mathbb{E}\left\{e^{\lambda_{nn}(\xi_1^{(n-1)} + \dots + \xi_i^{(n-1)})}; \overline{A_{n-1}(x)}, \bigcap_{k=0}^{n-2} \overline{A_k(x)}, Z_{n-1} = i\right\}. \end{aligned}$$

Note that the events $\bigcap_{k=0}^{n-2} \overline{A_k(x)}$ and $Z_{n-1} = i$ do not depend on the $\xi^{(n-1)}$'s. Therefore,

$$\begin{aligned} \mathbb{E}\left\{e^{\lambda_{nn} Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\right\} &= \sum_{i=1}^{\infty} \mathbb{E}\left\{e^{\lambda_{nn}(\xi_1^{(n-1)} + \dots + \xi_i^{(n-1)})}; \overline{A_{n-1}(x)}\right\} \mathbb{P}\left\{\bigcap_{k=0}^{n-2} \overline{A_k(x)}, Z_{n-1} = i\right\} \\ &= \sum_{i=1}^{\infty} \left(\mathbb{E}\{e^{\lambda_{nn} \xi}; \xi \leq m^n x(1 - \varepsilon)\}\right)^i \mathbb{P}\left\{\bigcap_{k=0}^{n-2} \overline{A_k(x)}, Z_{n-1} = i\right\}. \end{aligned}$$

If we put

$$\lambda_{n,n-1} := \log \mathbb{E}\{e^{\lambda_{nn} \xi}; \xi \leq m^n x(1 - \varepsilon)\},$$

then we receive a recursive equality

$$\mathbb{E}\left\{e^{\lambda_{nn}Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\right\} = \mathbb{E}\left\{e^{\lambda_{n,n-1}Z_{n-1}}; \bigcap_{k=0}^{n-2} \overline{A_k(x)}\right\}.$$

We iterate this recursion n times. Let us estimate $\lambda_{n,n-1}$ via λ_{nn} .

For every $z > 0$ and $y \leq z$, $e^y \leq 1 + y + y^2 e^z / 2$. Therefore,

$$\mathbb{E}\{e^{\lambda_{nn}\xi}; \xi \leq m^n x(1-\varepsilon)\} \leq 1 + \lambda_{nn}m + \lambda_{nn}^2 \mathbb{E}\{\xi^2; \xi \leq m^n x\} e^{\lambda_{nn}m^n x(1-\varepsilon)} / 2.$$

Since F is regularly varying with index -1 , for sufficiently large x ,

$$\mathbb{E}\{\xi^2; \xi \leq m^n x\} \leq \frac{3}{2}(m^n x)^2 \overline{F}(m^n x).$$

Hence,

$$\mathbb{E}\{e^{\lambda_{nn}\xi}; \xi \leq m^n x(1-\varepsilon)\} \leq 1 + \lambda_{nn}\left(m + \frac{3}{4}(\lambda_{nn}m^n x)m^n x \overline{F}(m^n x)e^{\lambda_{nn}m^n x(1-\varepsilon)}\right) \quad (48)$$

Denote

$$p_n(x) := \sum_{k=0}^{n-1} m^k \overline{F}(m^{k+1}x)$$

and make a special choice of initial λ_n :

$$\lambda_{nn} = \lambda_n(x) := (1 + \varepsilon) \frac{\log \frac{1}{p_n(x)x} - 2 \log \log \frac{1}{p_n(x)x}}{x \prod_{k=0}^{n-1} \left(m + \frac{m^{k+1}x \overline{F}(m^{k+1}x)}{(p_n(x)x)^{1-\varepsilon^2}}\right)}.$$

For the product, we have the following inequalities:

$$\begin{aligned} m^n \leq \prod_{k=0}^{n-1} \left(m + \frac{m^{k+1}x \overline{F}(m^{k+1}x)}{(p_n(x)x)^{1-\varepsilon^2}}\right) &= m^n \prod_{k=0}^{n-1} \left(1 + \frac{m^k \overline{F}(m^{k+1}x)}{p_n(x)} (p_n(x)x)^{\varepsilon^2}\right) \\ &\leq m^n e^{(p_n(x)x)^{\varepsilon^2} \sum_{k=0}^{n-1} \frac{m^k \overline{F}(m^{k+1}x)}{p_n(x)}} \\ &= m^n e^{(p_n(x)x)^{\varepsilon^2}}. \end{aligned}$$

Note that then this product is asymptotically equivalent to m^n because $p_n(x)x \rightarrow 0$.

Note also that then

$$\begin{aligned} \frac{1}{e^{\lambda_{nn}m^n x} - 1} &\leq \frac{1}{\left(\frac{1}{p_n(x)x}\right)^{1+\varepsilon+o(1)} \log^{-2(1+\varepsilon+o(1))} \frac{1}{p_n(x)x} - 1} \\ &\sim \frac{1}{(p_n(x)x)^{1+\varepsilon+o(1)} \log^{2(1+\varepsilon+o(1))} \frac{1}{p_n(x)x}} \\ &\leq c_1 (p_n(x)x)^{1+\varepsilon/2} \end{aligned} \quad (49)$$

ultimately in x uniformly in n . In particular, it goes to zero and the relation (46) follows.

Now estimate all λ_{nk} , $k \leq n-1$. With the choice of λ_{nn} made, it follows from (48) that

$$\begin{aligned} \mathbb{E}\{e^{\lambda_{nn}\xi}; \xi \leq m^n x(1-\varepsilon)\} &\leq 1 + \lambda_{nn} \left(m + \frac{3(1+\varepsilon)}{4} \log \frac{1}{p_n(x)x} \right. \\ &\quad \left. \times m^n x \bar{F}(m^n x) e^{(1-\varepsilon^2) \left(\log \frac{1}{p_n(x)x} - 2 \log \log \frac{1}{p_n(x)x} \right)} \right) \\ &\leq 1 + \lambda_{nn} \left(m + m^n x \bar{F}(m^n x) e^{(1-\varepsilon^2) \log \frac{1}{p_n(x)x}} \right), \end{aligned}$$

provided $1 + \varepsilon < 4/3$ and $2(1 - \varepsilon^2) > 1$. Thus,

$$\begin{aligned} \mathbb{E}\{e^{\lambda_{nn}\xi}; \xi \leq m^n x(1-\varepsilon)\} &\leq 1 + \lambda_{nn} \left(m + \frac{m^n x \bar{F}(m^n x)}{(p_n(x)x)^{1-\varepsilon^2}} \right) \\ &\leq \exp \left\{ \lambda_{nn} \left(m + \frac{m^n x \bar{F}(m^n x)}{(p_n(x)x)^{1-\varepsilon^2}} \right) \right\}, \end{aligned}$$

which yields

$$\lambda_{n,n-1} \leq \lambda_{nn} \left(m + \frac{m^n x \bar{F}(m^n x)}{(p_n(x)x)^{1-\varepsilon^2}} \right) = (1 + \varepsilon) \frac{\log \frac{1}{p_n(x)x} - 2 \log \log \frac{1}{p_n(x)x}}{x \prod_{k=0}^{n-2} \left(m + \frac{m^{k+1} x \bar{F}(m^{k+1} x)}{(p_n(x)x)^{1-\varepsilon^2}} \right)},$$

Iterating this estimate n times we finally deduce that

$$\mathbb{E}\left\{e^{\lambda_{nn} Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\right\} \leq \mathbb{E} e^{\lambda_{n0}} = e^{\lambda_{n0}} = \exp \left\{ \frac{1+\varepsilon}{x} \log \frac{1}{p_n(x)x} \right\}.$$

From here and (49),

$$\begin{aligned} \frac{\mathbb{E}\{e^{\lambda_{nn} Z_n}; \bigcap_{k=0}^{n-1} \overline{A_k(x)}\} - 1}{e^{\lambda_{nn} m^n x} - 1} &\leq c_2 x^{-1} (p_n(x)x)^{1+\varepsilon/2} \log \frac{1}{p_n(x)x} \\ &\leq c_3 x^{-1} (p_n(x)x)^{1+\varepsilon/4} \\ &= c_3 p_n(x) (p_n(x)x)^{\varepsilon/4} = o(p_n(x)) \end{aligned}$$

and (47) is also proven. This completes the proof of Theorem 4.

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